

Remarks on ϕ -coordinated modules for quantum vertex algebras

Haisheng Li

To James Lepowsky and Robert Wilson on Their 70th Birthdays

ABSTRACT. This paper is about ϕ -coordinated modules for weak quantum vertex algebras. Among the main results, several canonical connections among ϕ -coordinated modules for different ϕ are established. For vertex operator algebras, a reinterpretation of Frenkel-Huang-Lepowsky's theorem on contragredient module is given in terms of ϕ -coordinated modules.

1. Introduction

Partially motivated by Etingof-Kazhdan's notion of quantum vertex operator algebra (see [EK]), we have developed a theory of (weak) quantum vertex algebras (see [L3, L4, L6, L7]), where the notion of (weak) quantum vertex algebra generalizes the notions of vertex algebra, vertex super-algebra, and vertex color-algebra (see [X1], [X2]) in a certain natural way. To associate quantum vertex algebras to quantum affine algebras, in [L6] we developed a theory of ϕ -coordinated (quasi) modules for a weak quantum vertex algebra V , where ϕ is what we called an associate of the one-dimensional additive formal group (law) $F_{\text{add}}(x, y) = x + y$.

By definition (cf. [H]), a general one-dimensional *formal group* (over \mathbb{C}) is a formal power series $F(x, y) \in \mathbb{C}[[x, y]]$, satisfying the conditions

$$F(x, 0) = x, \quad F(0, y) = y, \quad F(F(x, y), z) = F(x, F(y, z)).$$

As it was pointed out by Borcherds (see [B]), the formal group law $F_{\text{add}}(x, y)$ underlies the theory of vertex algebras and modules. This is more apparent from the associativity:

$$Y(u, z + x)Y(v, x) = Y(Y(u, z)v, x).$$

This is also the case for the theory of (weak) quantum vertex algebras and modules, in which the same associativity is postulated though the usual locality (commutativity) is generalized to a braided locality. Moreover, formal group law $F_{\text{add}}(x, y)$ also underlies the theory of twisted modules (see [FLM], [FFR], [D], [X1], [L1], [LTW]).

1991 *Mathematics Subject Classification.* Primary 17B69, 17B65; Secondary 17B10, 81R10.

Key words and phrases. Quantum vertex algebra, ϕ -coordinated module, formal group, associate.

The author was supported in part by China NSF Grants #11471268, 11571391.

An *associate* of $F_{\text{add}}(x, y)$ (see [L6]) is defined to be a formal series $\phi(x, z) \in \mathbb{C}((x))[[z]]$, satisfying the conditions

$$\phi(x, 0) = x, \quad \phi(\phi(x, y), z) = \phi(x, y + z) \quad (= \phi(x, F_{\text{add}}(y, z))).$$

Philosophically speaking, an associate to $F_{\text{add}}(x, y)$ is the same as a G -set to a group G . From the definition, the formal group $F_{\text{add}}(x, y)$ itself is an associate. It was proved in [L6] that for every $p(x) \in \mathbb{C}((x))$, $e^{zp(x)\frac{d}{dx}}(x)$ is an associate of $F_{\text{add}}(x, y)$ and every associate can be obtained this way with $p(x)$ uniquely determined. In particular, taking $p(x) = x^{n+1}$ for $n \in \mathbb{Z}$, we obtain associates

$$(1.1) \quad \phi_n(x, z) = e^{zx^{n+1}\frac{d}{dx}}(x) = \begin{cases} xe^z & \text{for } n = 0 \\ (x^{-n} - nz)^{-1/n} & \text{for } n \neq 0 \end{cases}$$

(cf. [FHL]; (2.6.5)), where

$$\phi_{-1}(x, z) = e^{z\frac{d}{dx}}(x) = x + z = F_{\text{add}}(x, z).$$

The essence of [L6] is to attach a theory of ϕ -coordinated modules for any weak quantum vertex algebra V to each associate $\phi(x, z)$ of $F_{\text{add}}(x, y)$, where the defining associativity of a ϕ -coordinated V -module (W, Y_W) can be roughly described as

$$(1.2) \quad Y_W(u, x_1)Y_W(v, x)|_{x_1=\phi(x, z)} = Y_W(Y(u, z)v, x).$$

To associate quantum vertex algebras to quantum affine algebras, we have mostly focused on ϕ -coordinated quasi modules with $\phi(x, z) = xe^z$ (see [L6, L7]). Indeed, by using this very theory, weak quantum vertex algebras have been successfully associated to quantum affine algebras *conceptually*, while the construction of an explicit association is in progress. On the other hand, let V be a vertex operator algebra, (W, Y_W) a V -module. For $v \in V$, set $X_W(v, x) = Y_W(x^{L(0)}v, x)$. It was shown in [L5] that (W, X_W) carries the structure of a ϕ -coordinated module with $\phi(x, z) = xe^z$ for the new vertex operator algebra obtained by Zhu (see [Z1, Z2]) on V . This result is closely related to a previous result of Lepowsky (see [Le]). Further investigation on ϕ -coordinated quasi modules with $\phi(x, z) = xe^z$ for weak quantum vertex algebras is yet to be done.

In this paper, we explore canonical connections among ϕ -coordinated modules for a general weak quantum vertex algebra for various associates ϕ . Let $\phi(x, z)$ be an associate of $F_{\text{add}}(x, y)$ and let $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$. Set $\phi_f(x, z) = f^{-1}(\phi(f(x), z))$. We show that $\phi_f(x, z)$ is also an associate of $F_{\text{add}}(x, y)$ and that for any weak quantum vertex algebra V , the category of ϕ -coordinated V -modules is canonically isomorphic to the category of ϕ_f -coordinated V -modules. Furthermore, we show that an associate $\phi(x, z)$ satisfies the condition that $\phi_f(x, z) = x + z = F_{\text{add}}(x, z)$ for some $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$ if and only if $\phi(x, z) = e^{zp(x)\frac{d}{dx}}(x)$ for $p(x) \in \mathbb{C}[[x]]$ with $p(0) \neq 0$.

In vertex operator algebra theory, contragredient module due to Frenkel-Huang-Lepowsky (see [FHL]) plays a fundamental role. For any module (W, Y_W) over a vertex operator algebra V , the underlying space of the contragredient module is the restricted dual $W' = \oplus_{h \in \mathbb{C}} W_{(h)}^*$ of W , where the vertex operator map Y'_W is defined by

$$\langle Y'_W(v, x)\alpha, w \rangle = \langle \alpha, Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x^{-1})w \rangle$$

for $v \in V$, $\alpha \in W'$, $w \in W$. In this theory, the Virasoro algebra, which is part of the vertex operator algebra structure of V , is essential. A natural question is what

one can possibly get if V is a weak quantum vertex algebra, in particular, a general vertex algebra without a conformal vector.

Note that the contragredient module theorem of [FHL] has been generalized by Huang-Lepowsky-Zhang (see [HLZ1, HLZ2]) to strongly graded conformal vertex algebras.

Let V be a weak quantum vertex algebra, (W, Y_W) a V -module. Consider the full dual space W^* . For $v \in V$, we define $Y_W^*(v, x) \in (\text{End } W^*)[[x, x^{-1}]]$ by

$$\langle Y_W^*(v, x)\alpha, w \rangle = \langle \alpha, Y_W(v, x^{-1})w \rangle$$

for $\alpha \in W^*$, $w \in W$. Furthermore, we define $D(W)$ to be the subspace of W^* , consisting of each α such that

$$Y_W^*(v, x)\alpha \in W^*((x)) \quad \text{for all } v \in V.$$

It is proved in this paper that $D(W)$ equipped with the vertex operator map Y_W^* is a ϕ -coordinated module with $\phi(x, z) = \phi_1(x, -z) = \frac{x}{1+zx}$ for the opposite algebra V^o (see [S]; cf. [BK]), where the vacuum vector is the same as that of V and the vertex operator map $Y^o(\cdot, x)$ is defined by

$$Y^o(u, x)v = e^{x\mathcal{D}}Y(v, -x)u \quad \text{for } u, v \in V.$$

Note that when V is a vertex operator algebra, for any V -module W a subspace of W^* , which was also denoted by $D(W)$, was introduced in [L2] and it was proved that $D(W)$ with the vertex operator map Y_W' is a weak V -module in the sense that $D(W)$ is a module for V viewed as a vertex algebra. This weak V -module structure was used therein to study certain finiteness properties of W . In fact, in this case the two definitions give the same space $D(W)$ though the vertex operator maps differ by a twist. In this paper, as a reinterpretation of the contragredient module theorem of [FHL], we show that if V is a vertex operator algebra, the category of modules for V viewed as a vertex algebra is canonically isomorphic to the category of ϕ -coordinated V -modules with $\phi(x, z) = \frac{x}{1+zx}$.

This paper is organized as follows: In Section 2, we recall the basic results on associates of $F_{\text{add}}(x, y)$ and on ϕ -coordinated modules for a weak quantum vertex algebra, and we show that the category of ϕ -coordinated V -modules is isomorphic to the category of ϕ_f -coordinated V -modules. In Section 3, we study the dual $D(W)$ of a module W for a weak quantum vertex algebra V and we give the equivalence between the categories of modules for V viewed as a vertex algebra and ϕ -coordinated V -modules with $\phi(x, z) = \frac{x}{1+zx}$ for any vertex operator algebra V .

2. ϕ -coordinated modules for weak quantum vertex algebras

In this section, we first recall the notion of associate for the one-dimensional additive formal group (law) and the notion of ϕ -coordinated module for a weak quantum vertex algebra V with respect to an associate ϕ , and we then show that the category of ϕ -coordinated V -modules is isomorphic to the category of ϕ_f -coordinated V -modules..

We begin with the one-dimensional additive formal group (law), which is the formal power series

$$F_{\text{add}}(x, y) = x + y \in \mathbb{C}[[x, y]].$$

(See for example [H] for the definition of a one-dimensional formal group.)

The following notion was introduced in [L6]:

DEFINITION 2.1. An *associate* of $F_{\text{add}}(x, y)$ is a formal series $\phi(x, z) \in \mathbb{C}((x))[[z]]$ satisfying the conditions

$$(2.1) \quad \phi(x, 0) = x, \quad \phi(\phi(x, y), z) = \phi(x, y + z).$$

Note that every associate $\phi(x, z)$ is invertible in $\mathbb{C}((x))[[z]]$ as $\phi(x, 0) = x$ is an invertible element of $\mathbb{C}((x))$. Then $\phi(x, z)^n$ is a well defined element of $\mathbb{C}((x))[[z]]$ for every integer n .

The following result was also obtained in [L6]:

PROPOSITION 2.2. For $p(x) \in \mathbb{C}((x))$, set

$$\phi_{p(x)}(x, z) = e^{zp(x)\frac{d}{dx}}x \in \mathbb{C}((x))[[z]].$$

Then $\phi_{p(x)}(x, z)$ is an associate of $F_{\text{add}}(x, y)$. On the other hand, every associate is of this form with $p(x)$ uniquely determined.

For $n \in \mathbb{Z}$, set

$$(2.2) \quad \phi_n(x, z) = e^{zx^{n+1}\frac{d}{dx}} \cdot x \in \mathbb{C}((x))[[z]].$$

(Note that the centerless Virasoro algebra, namely the Witt algebra, is given by $L(n) = -x^{n+1}\frac{d}{dx}$ for $n \in \mathbb{Z}$.) In particular, we have

$$(2.3) \quad \phi_{-1}(x, z) = x + z = F(x, z), \quad \phi_0(x, z) = xe^z, \quad \phi_1(x, z) = \frac{x}{1 - zx},$$

where as a convention, $\frac{x}{1 - zx}$ stands for the formal power series expansion in the nonnegative powers of z .

REMARK 2.3. Let $\phi(x, z)$ be an associate of $F_{\text{add}}(x, y)$ and let

$$A(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$$

with W a general vector space. We have

$$(2.4) \quad A(\phi(x_2, z), x_2) = A(x_1, x_2)|_{x_1=\phi(x_2, z)} \in (\text{Hom}(W, W((x_2)))) [[z]].$$

On the other hand, for $B(x, z) \in (\text{Hom}(W, W((x)))) [[z]]$, we have

$$(2.5) \quad B(\phi(x_2, -z), z) = B(x_1, z)|_{x_1=\phi(x_2, -z)} \in (\text{Hom}(W, W((x_2)))) [[z]].$$

Furthermore, for $A(x_1, x_2) \in \text{Hom}(W, W((x_1, x_2)))$, we have

$$(2.6) \quad (A(x_1, x_2)|_{x_1=\phi(x_2, z)})|_{x_2=\phi(x_1, -z)} = A(x_1, x_2),$$

noticing that

$$\phi(x_2, z)|_{x_2=\phi(x_1, -z)} = \phi(\phi(x_1, -z), z) = \phi(x_1, (-z) + z) = \phi(x_1, 0) = x_1.$$

Let $\phi(x, z) = e^{zp(x)\frac{d}{dx}} \cdot x$ with $p(x) \in \mathbb{C}[x, x^{-1}]$. Noticing that $\phi(x, z) \in \mathbb{C}[x, x^{-1}][[z]]$, we have $\phi(x^{-1}, z) \in \mathbb{C}[x, x^{-1}][[z]]$. As $\phi(x^{-1}, 0) = x^{-1}$, $\phi(x^{-1}, z)$ is an invertible element of $\mathbb{C}((x))[[z]]$. Since

$$\phi(x^{-1}, z) = e^{-zp(1/x)x^2\frac{d}{dx}} \cdot x^{-1},$$

$$1 = e^{-zp(1/x)x^2\frac{d}{dx}} \cdot (x \cdot x^{-1}) = \left(e^{-zp(1/x)x^2\frac{d}{dx}} \cdot x \right) \left(e^{-zp(1/x)x^2\frac{d}{dx}} \cdot x^{-1} \right),$$

we get

$$\frac{1}{\phi(x^{-1}, z)} = e^{-zp(1/x)x^2\frac{d}{dx}} \cdot x,$$

which is also an associate of $F_{\text{add}}(x, y)$. Thus we have proved:

LEMMA 2.4. Suppose $\phi(x, z) = e^{zp(x)\frac{d}{dx}} \cdot x$ with $p(x) \in \mathbb{C}[x, x^{-1}]$. Define

$$(2.7) \quad \phi^*(x, z) = \frac{1}{\phi(x^{-1}, z)} \in \mathbb{C}((x))[[z]].$$

Then $\phi^*(x, z)$ is also an associate of $F_{\text{add}}(x, y)$ and we have

$$(2.8) \quad \phi^*(x, z) = e^{-zp(1/x)x^2\frac{d}{dx}} \cdot x.$$

It can be readily seen that $(\phi^*)^* = \phi$ for $\phi(x, z) = e^{zp(x)\frac{d}{dx}} \cdot x$ with $p(x) \in \mathbb{C}[x, x^{-1}]$.

REMARK 2.5. Let $p(x) = x^{n+1}$ with $n \in \mathbb{Z}$. We have $-x^2p(1/x) = -x^{1-n}$. Then

$$(2.9) \quad \phi_n^*(x, z) = \phi_{-n}(x, -z) \quad \text{for } n \in \mathbb{Z}.$$

In particular, we have $\phi_{-1}^*(x, z) = \phi_1(x, -z) = \frac{x}{1+zx}$ and $\phi_0^*(x, z) = \phi_0(x, -z)$.

Next, we recall from [L3] (cf. [EK]) the notion of weak quantum vertex algebra, which generalizes the notions of vertex algebra, vertex super-algebra, and vertex color-algebra (see [X1], [X2]) in a certain natural way.

DEFINITION 2.6. A *weak quantum vertex algebra* is a vector space V equipped with a linear map

$$\begin{aligned} Y(\cdot, x) : V &\rightarrow (\text{End} V)[[x, x^{-1}]], \\ v &\mapsto Y(v, x) = \sum_{n \in \mathbb{Z}} v_n x^{-n-1} \quad (\text{where } v_n \in \text{End} V) \end{aligned}$$

and equipped with a distinguished vector $\mathbf{1} \in V$, satisfying the conditions that

$$Y(u, x)v \in V((x)) \quad \text{for } u, v \in V,$$

$$Y(\mathbf{1}, x)v = v, \quad Y(u, x)\mathbf{1} \in V[[x]] \quad \text{and} \quad \lim_{x \rightarrow 0} Y(u, x)\mathbf{1} = u,$$

and that for $u, v \in V$, there exist (finitely many)

$$u^{(i)}, v^{(i)} \in V, \quad f_i(x) \in \mathbb{C}((x)) \quad (i = 1, \dots, r)$$

such that

$$\begin{aligned} &x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y(u, x_1) Y(v, x_2) \\ &\quad - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \sum_i f_i(x_2 - x_1) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1) \\ (2.10) \quad &= x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y(Y(u, x_0)v, x_2). \end{aligned}$$

By the standard formal variable techniques we immediately have (cf. [L3]):

PROPOSITION 2.7. In Definition 2.6, the \mathcal{S} -Jacobi identity (2.10) is equivalent to the property that there exists a nonnegative integer k such that

$$(2.11) \quad (x_1 - x_2)^k Y(u, x_1) Y(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^r f_i(x_2 - x_1) Y(v^{(i)}, x_2) Y(u^{(i)}, x_1),$$

$$(2.12) \quad ((x_1 - x_2)^k Y(u, x_1) Y(v, x_2))|_{x_1=x_2+x_0} = x_0^k Y(Y(u, x_0)v, x_2).$$

Let V be a weak quantum vertex algebra. Denote by \mathcal{D} the linear operator defined by

$$(2.13) \quad \mathcal{D}(v) = \lim_{x \rightarrow 0} \frac{d}{dx} Y(v, x) \mathbf{1} \quad \text{for } v \in V.$$

Then (see [L3])

$$(2.14) \quad [\mathcal{D}, Y(v, x)] = Y(\mathcal{D}v, x) = \frac{d}{dx} Y(v, x).$$

Furthermore, for $u, v \in V$, we have (see [EK])

$$(2.15) \quad Y(u, x)v = e^{x\mathcal{D}} \sum_{i=1}^r f_i(-x) Y(v^{(i)}, -x) u^{(i)},$$

where $u^{(i)}, v^{(i)} \in V$, $f_i(x) \in \mathbb{C}((x))$ such that (2.11) holds.

REMARK 2.8. Let V be a weak quantum vertex algebra. Define a linear map

$$(2.16) \quad Y^o(\cdot, x) : V \rightarrow (\text{End}V)[[x, x^{-1}]]$$

by

$$(2.17) \quad Y^o(u, x)v = e^{x\mathcal{D}} Y(v, -x)u \quad \text{for } u, v \in V.$$

Then $(V, Y^o, \mathbf{1})$ is a weak quantum vertex algebra (see [S]; cf. [BK]). Notice that if V is a vertex algebra, then $(V, Y^o, \mathbf{1})$ coincides with $(V, Y, \mathbf{1})$ as

$$Y(u, x)v = e^{x\mathcal{D}} Y(v, -x)u = Y^o(u, x)v \quad \text{for all } u, v \in V.$$

Note that the last condition in Definition 2.6 amounts to that there is a linear map

$$\mathcal{S}(x) : V \otimes V \rightarrow V \otimes V \otimes \mathbb{C}((x))$$

such that for $u, v \in V$, (2.10) holds with

$$\mathcal{S}(x)(v \otimes u) = \sum_{i=1}^r v^{(i)} \otimes u^{(i)} \otimes f_i(x).$$

By a *weak quantum vertex algebra with a constant map* \mathcal{S} we mean that there is a constant linear map \mathcal{S} such that (2.10) holds. Note that here \mathcal{S} does not depend on x , but it could depend on vectors $u, v \in V$.

The following notion was introduced in [L6]:

DEFINITION 2.9. Let V be a weak quantum vertex algebra, ϕ an associate of $F_{\text{add}}(x, y)$. A ϕ -coordinated V -module is a vector space W equipped with a linear map

$$\begin{aligned} Y_W(\cdot, x) : V &\rightarrow (\text{End}W)[[x, x^{-1}]], \\ v &\mapsto Y_W(v, x) \end{aligned}$$

satisfying the conditions that

$$Y_W(v, x)w \in W((x)) \quad \text{for } v \in V, w \in W,$$

$Y_W(\mathbf{1}, x) = 1_W$, and that for any $u, v \in V$, there exists $k \in \mathbb{N}$ such that

$$\begin{aligned} (x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) &\in \text{Hom}(W, W((x_1, x_2))), \\ ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2))|_{x_1=\phi(x_2, z)} &= (\phi(x_2, z) - x_2)^k Y_W(Y(u, z)v, x_2). \end{aligned} \quad (2.18)$$

Note that in Definition 2.9, the first part of the second condition guarantees the existence of the substitution on the left-hand side of (2.18). The following was proved in [L6]:

LEMMA 2.10. *Let V be a weak quantum vertex algebra and let (W, Y_W) be a ϕ -coordinated V -module. Then*

$$(2.19) \quad Y_W(e^{z\mathcal{D}}u, x) = Y_W(u, \phi(x, z)) \quad \text{for } u \in V.$$

REMARK 2.11. In case $\phi(x, z) = x + z$ (the formal group itself), the notion of ϕ -coordinated V -module is equivalent to the notion of V -module defined in [L3]. If (W, Y_W) is a V -module, then for $u, v \in V$,

$$(2.20) \quad \begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(u, x_1) Y_W(v, x_2) \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) \sum_i f_i(x_2 - x_1) Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1) \\ & = x_2^{-1} \delta \left(\frac{x_1 - x_0}{x_2} \right) Y_W(Y(u, x_0)v, x_2), \end{aligned}$$

where $u^{(i)}, v^{(i)}, f_i(x)$ are given as in (2.10). It was proved therein that

$$(2.21) \quad Y_W(\mathcal{D}v, x) = \frac{d}{dx} Y_W(v, x) \quad \text{for } v \in V.$$

Here, we have:

PROPOSITION 2.12. *Let V be a weak quantum vertex algebra with a constant map \mathcal{S} , let $\phi(x, z)$ be an associate of $F_{\text{add}}(x, y)$, and let (W, Y_W) be a ϕ -coordinated V -module. Then for any $u, v \in V$, there exists a nonnegative integer k such that*

$$(2.22) \quad (x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^r Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1),$$

where $u^{(i)}, v^{(i)} \in V$ such that $Y(u, x)v = \sum_{i=1}^r e^{x\mathcal{D}}Y(v^{(i)}, -x)u^{(i)}$.

PROOF. Let $u, v \in V$. Since \mathcal{S} is constant, we have $Y(u, x)v = \sum_{i=1}^r e^{x\mathcal{D}}Y(v^{(i)}, -x)u^{(i)}$ for some $u^{(i)}, v^{(i)} \in V$. By definition, there exists a nonnegative integer k such that

$$\begin{aligned} & (x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))), \\ & ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2))|_{x_1=\phi(x_2, z)} = (\phi(x_2, z) - x_2)^k Y_W(Y(u, z)v, x_2), \end{aligned}$$

and such that

$$\begin{aligned} & (x_1 - x_2)^k Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1) \in \text{Hom}(W, W((x_1, x_2))), \\ & \left((x_1 - x_2)^k Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1) \right)|_{x_2=\phi(x_1, -z)} \\ & = (x_1 - \phi(x_1, -z))^k Y_W(Y(v^{(i)}, -z)u^{(i)}, x_1) \end{aligned}$$

for $i = 1, \dots, r$. Noticing that

$$\sum_{i=1}^r Y_W(Y(v^{(i)}, -z)u^{(i)}, x_1) = Y_W(e^{-z\mathcal{D}}Y(u, z)v, x_1) = Y_W(Y(u, z)v, \phi(x_1, -z)),$$

we get

$$\begin{aligned}
& \left((x_1 - x_2)^k \sum_{i=1}^r Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1) \right) \Big|_{x_2=\phi(x_1, -z)} \\
&= (x_1 - \phi(x_1, -z))^k Y_W(Y(u, z)v, \phi(x_1, -z)) \\
&= ((\phi(x_2, z) - x_2)^k Y_W(Y(u, z)v, x_2)) \Big|_{x_2=\phi(x_1, -z)} \\
&= (((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)) \Big|_{x_1=\phi(x_2, z)}) \Big|_{x_2=\phi(x_1, -z)} \\
&= (x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2),
\end{aligned}$$

where we are using the fact that $\phi(\phi(x_1, -z), z) = x_1$ (recalling Remark 2.3). From this we obtain weak commutativity relation

$$(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^r Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1),$$

as desired. \square

Next, we show that for some associate ϕ , the category of ϕ -coordinated V -modules is actually isomorphic to the category of V -modules canonically. Note that for any $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$, $f(x)$ is invertible with respect to the composition, i.e., there exists $f^{-1}(x) \in x\mathbb{C}[[x]]$ such that $f(f^{-1}(x)) = x = f^{-1}(f(x))$.

LEMMA 2.13. *Let $\phi(x, z)$ be an associate of $F_{\text{add}}(x, y)$ and let $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$. Define*

$$\phi_f(x, z) = f^{-1}(\phi(f(x), z)).$$

Then $\phi_f(x, z)$ is an associate of $F_{\text{add}}(x, y)$.

PROOF. It is straightforward: As $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$, we have $f^{-1}(z) \in z\mathbb{C}[[z]]$ and $f(x)^n \in x^n\mathbb{C}[[x]]$ for $n \in \mathbb{Z}$. Thus $\phi(f(x), z) \in \mathbb{C}((x))[[z]]$ and then $f^{-1}(\phi(f(x), z)) \in \mathbb{C}((x))[[z]]$. Furthermore, we have

$$\begin{aligned}
\phi_f(x, 0) &= f^{-1}(\phi(f(x), 0)) = f^{-1}(f(x)) = x, \\
\phi_f(\phi_f(x, y), z) &= f^{-1}(\phi(f(\phi_f(x, y)), z)) \\
&= f^{-1}(\phi(\phi(f(x), y), z)) = f^{-1}(\phi(f(x), y + z)) = \phi_f(x, y + z).
\end{aligned}$$

This proves that $\phi_f(x, z)$ is an associate of $F_{\text{add}}(x, y)$. \square

PROPOSITION 2.14. *Let V be a weak quantum vertex algebra, let $\phi(x, z)$ be an associate of $F_{\text{add}}(x, y)$, and let $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$. Let (W, Y_W) be a ϕ -coordinated V -module. For $v \in V$, set*

$$(2.23) \quad Y_W^f(v, x) = Y_W(v, f(x)).$$

Then (W, Y_W^f) carries the structure of a ϕ_f -coordinated V -module where $\phi_f(x, z)$ is defined as in Lemma 2.13.

PROOF. From definition we have $Y_W^f(\mathbf{1}, x) = Y_W(\mathbf{1}, f(x)) = 1_W$ and

$$Y_W^f(v, x)w = Y_W(v, f(x))w \in W((x)) \quad \text{for } v \in V, w \in W.$$

Let $u, v \in V$. There exists a nonnegative integer k such that

$$(z_1 - z_2)^k Y_W(u, z_1) Y_W(v, z_2) \in \text{Hom}(W, W((z_1, z_2)))$$

and

$$\left((z_1 - z_2)^k Y_W(u, z_1) Y_W(v, z_2) \right) |_{z_1=\phi(z_2, x_0)} = (\phi(z_2, x_0) - z_2)^k Y_W(Y(u, x_0)v, z_2).$$

Then

$$(f(x_1) - f(x_2))^k Y_W(u, f(x_1)) Y_W(v, f(x_2)) \in \text{Hom}(W, W((x_1, x_2))).$$

As $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$, we have

$$f(x_1) - f(x_2) = (x_1 - x_2)g(x_1, x_2),$$

where $g(x_1, x_2) \in \mathbb{C}[[x_1, x_2]]$ with $g(0, 0) = f'(0)$. Since $g(0, 0) \neq 0$, $g(x_1, x_2)$ is an invertible element of $\mathbb{C}[[x_1, x_2]]$. Then we get

$$(x_1 - x_2)^k Y_W(u, f(x_1)) Y_W(v, f(x_2)) \in \text{Hom}(W, W((x_1, x_2))).$$

The substitution $x_1 = \phi_f(x_2, x_0) = f^{-1}(\phi(f(x_2), x_0))$ amounts to the substitution $f(x_1) = \phi(f(x_2), x_0)$. Thus

$$\begin{aligned} & g(\phi_f(x_2, x_0), x_2)^k \left((x_1 - x_2)^k Y_W(u, f(x_1)) Y_W(v, f(x_2)) \right) |_{x_1=\phi_f(x_2, x_0)} \\ &= \left((f(x_1) - f(x_2))^k Y_W(u, f(x_1)) Y_W(v, f(x_2)) \right) |_{f(x_1)=\phi(f(x_2), x_0)} \\ &= (\phi(f(x_2), x_0) - f(x_2))^k Y_W(Y(u, x_0)v, f(x_2)) \\ &= (f(\phi_f(x_2, x_0)) - f(x_2))^k Y_W(Y(u, x_0)v, f(x_2)) \\ &= g(\phi_f(x_2, x_0), x_2)^k (\phi_f(x_2, x_0) - x_2)^k Y_W(Y(u, x_0)v, f(x_2)), \end{aligned}$$

which implies

$$\begin{aligned} & \left((x_1 - x_2)^k Y_W(u, f(x_1)) Y_W(v, f(x_2)) \right) |_{x_1=\phi_f(x_2, x_0)} \\ &= (\phi_f(x_2, x_0) - x_2)^k Y_W(Y(u, x_0)v, f(x_2)). \end{aligned}$$

This proves that (W, Y_W^f) carries the structure of a ϕ_f -coordinated V -module. \square

Furthermore, it can be readily seen that if θ is a homomorphism of ϕ -coordinated V -modules from (W_1, Y_{W_1}) to (W_2, Y_{W_2}) , then θ is also a homomorphism of ϕ_f -coordinated V -modules from $(W_1, Y_{W_1}^f)$ to $(W_2, Y_{W_2}^f)$. To summarize we have:

THEOREM 2.15. *Let V be a weak quantum vertex algebra, let $\phi(x, z)$ be an associate of $F_{\text{add}}(x, y)$, and let $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$. Then the category of ϕ -coordinated V -modules is isomorphic to the category of ϕ_f -coordinated V -modules canonically.*

Set $G = \{f(z) \in z\mathbb{C}[[z]] \mid f'(0) \neq 0\}$. Then G is a group with respect to the composition. It is straightforward to see that Lemma 2.13 defines a right action of group G on the set of associates of $F_{\text{add}}(x, y)$. This gives rise to an equivalence relation on the set of associates of $F_{\text{add}}(x, y)$.

LEMMA 2.16. *Let $\phi(x, z) = e^{zp(x)\frac{d}{dx}}(x)$ with $p(x) \in \mathbb{C}((x))$. Then $\phi(x, z)$ is equivalent to $F_{\text{add}}(x, z)$ if and only if $p(x) \in \mathbb{C}[[x]]$ with $p(0) \neq 0$.*

PROOF. By definition, $\phi(x, z)$ is equivalent to $F_{\text{add}}(x, z)$ if and only if there exists $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$ such that $\phi(x, z) = f^{-1}(f(x) + z)$, or equivalently, $f(\phi(x, z)) = f(x) + z$. Noticing that

$$f(\phi(x, z)) = f\left(e^{zp(x)\frac{d}{dx}}(x)\right) = e^{zp(x)\frac{d}{dx}}f(x),$$

we see that $f(\phi(x, z)) = f(x) + z$ if and only if $p(x)\frac{d}{dx}f(x) = 1$, i.e., $p(x)f'(x) = 1$. It is clear that there exists $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$ such that $p(x)f'(x) = 1$ if and only if $p(x) \in \mathbb{C}[[x]]$ with $p(0) \neq 0$. \square

REMARK 2.17. It follows from Lemma 2.16 that $\phi_n(x, z)$ for $n \neq -1$ is not equivalent to $F_{\text{add}}(x, z) = x + z$. In particular, $\phi_0(x, z) = xe^z$ is *not* equivalent to $F_{\text{add}}(x, z)$.

REMARK 2.18. Let $p_1(x), p_2(x) \in \mathbb{C}((x))$. One can show that $\phi_{p_1(x)}(x, z)$ is equivalent to $\phi_{p_2(x)}(x, z)$ if and only if $p_1(x)f'(x) = p_2(x)f'(x)$ for some $f(x) \in x\mathbb{C}[[x]]$ with $f'(0) \neq 0$. But, we are unable to get an explicit characterization.

3. ϕ -coordinated module $(D(W), Y_W^*)$

In this section, given any module (W, Y_W) for a weak quantum vertex algebra V with a constant map \mathcal{S} , we construct a ϕ -coordinated V -module $(D(W), Y_W^*)$ out of W^* with $\phi(x, z) = \frac{x}{1+zx}$. We also show that for a vertex operator algebra V , the categories of ϕ -coordinated V -modules and modules for V viewed as a vertex algebra are isomorphic. Note that a V -module is the same as a ϕ_{-1} -coordinated V -module while $\frac{x}{1+zx} = \phi_1(x, -z) = \phi_{-1}^*(x, z)$.

First, we show that a certain Jacobi identity holds on ϕ -coordinated V -modules with $\phi(x, z) = \frac{x}{1+zx}$ just as with usual V -modules. (A Jacobi identity on ϕ -coordinated modules with $\phi(x, z) = xe^z$ was obtained in [L6].)

PROPOSITION 3.1. *Let V be a weak quantum vertex algebra with a constant map \mathcal{S} and let $\phi(x, z) = \phi_1(x, -z) = \frac{x}{1+zx}$. Then, in the definition of a ϕ -coordinated V -module, the second condition is equivalent to*

$$\begin{aligned}
 & z^{-1}\delta\left(\frac{x_2^{-1} - x_1^{-1}}{z}\right)Y_W(u, x_1)Y_W(v, x_2) \\
 & \quad - z^{-1}\delta\left(\frac{x_1^{-1} - x_2^{-1}}{-z}\right)\sum_{i=1}^r Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1) \\
 (3.1) \quad & = x_2\delta\left(\frac{x_1^{-1} + z}{x_2^{-1}}\right)Y_W(Y(u, -z)v, x_2)
 \end{aligned}$$

for $u, v \in V$, where $u^{(i)}, v^{(i)} \in V$ such that $Y(u, x)v = \sum_{i=1}^r e^{x\mathcal{D}}Y(v^{(i)}, -x)u^{(i)}$.

PROOF. First, assume that (3.1) holds for any $u, v \in V$. Let $u, v \in V$. There exists a nonnegative integer k such that $x^k Y(u, x)v \in V[[x]]$. Applying $\text{Res}_z z^k x_1^k x_2^k$ to (3.1) we get

$$(3.2) \quad (x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^r Y_W(v^{(i)}, x_2)Y_W(u^{(i)}, x_1),$$

which implies

$$(x_1 - x_2)^k Y_W(u, x_1)Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2))).$$

Multiplying both sides of (3.1) by $z^k x_1^k x_2^k$, and then using (3.2) and the basic delta-function properties we get

$$\begin{aligned} & x_2 \delta \left(\frac{x_1^{-1} + z}{x_2^{-1}} \right) ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)) \\ = & x_2 \delta \left(\frac{x_1^{-1} + z}{x_2^{-1}} \right) (zx_1 x_2)^k Y_W(Y(u, -z)v, x_2), \end{aligned}$$

which can also be written as

$$\begin{aligned} & x_1 \delta \left(\frac{x_2^{-1} - z}{x_1^{-1}} \right) ((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)) \\ = & x_1 \delta \left(\frac{x_2^{-1} - z}{x_1^{-1}} \right) (zx_1 x_2)^k Y_W(Y(u, -z)v, x_2). \end{aligned}$$

Then applying $\text{Res}_{x_1} x_1^{-2}$ we get

$$((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)) \big|_{x_1^{-1} = x_2^{-1} - z} = \left(\frac{zx_2}{x_2^{-1} - z} \right)^k Y_W(Y(u, -z)v, x_2).$$

That is,

$$((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)) \big|_{x_1 = \frac{x_2}{1 - zx_2}} = \left(\frac{x_2}{1 - zx_2} - x_2 \right)^k Y_W(Y(u, -z)v, x_2).$$

Thus

$$((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)) \big|_{x_1 = \frac{x_2}{1 + zx_2}} = \left(\frac{x_2}{1 + zx_2} - x_2 \right)^k Y_W(Y(u, z)v, x_2).$$

This proves that (W, Y_W) is a ϕ -coordinated V -module.

On the other hand, assume that (W, Y_W) is a ϕ -coordinated V -module. Let $u, v \in V$, and let $u^{(i)}, v^{(i)} \in V$, $1 \leq i \leq r$ such that $Y(u, x)v = \sum_{i=1}^r e^{x\mathcal{D}} Y(v^{(i)}, -x)u^{(i)}$. By Proposition 2.12, there exists a nonnegative integer k such that

$$(3.3) \quad (x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) = (x_1 - x_2)^k \sum_{i=1}^r Y_W(v^{(i)}, x_2) Y_W(u^{(i)}, x_1).$$

As this implies $(x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2) \in \text{Hom}(W, W((x_1, x_2)))$, we have

$$((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)) \big|_{x_1 = \phi(x_2, z)} = \left(\frac{x_2}{1 + zx_2} - x_2 \right)^k Y_W(Y(u, z)v, x_2),$$

which gives

$$((x_1 - x_2)^k Y_W(u, x_1) Y_W(v, x_2)) \big|_{x_1^{-1} = x_2^{-1} - z} = \left(\frac{zx_2}{x_2^{-1} - z} \right)^k Y_W(Y(u, -z)v, x_2).$$

Note that

$$z^{-1} \delta \left(\frac{x_2^{-1} - x_1^{-1}}{z} \right) - z^{-1} \delta \left(\frac{x_1^{-1} - x_2^{-1}}{-z} \right) = x_2 \delta \left(\frac{x_1^{-1} + z}{x_2^{-1}} \right).$$

Then Jacobi identity (3.1) follows. \square

DEFINITION 3.2. Let V be a weak quantum vertex algebra and let (W, Y_W) be a V -module. Define a linear map

$$\begin{aligned} Y_W^*(\cdot, x) : V &\rightarrow (\text{End } W^*)[[x, x^{-1}]], \\ v &\mapsto Y_W^*(v, x) \end{aligned}$$

by

$$(3.4) \quad \langle Y_W^*(v, x)\alpha, w \rangle = \langle \alpha, Y_W(v, x^{-1})w \rangle$$

for $v \in V$, $\alpha \in W^*$, $w \in W$.

LEMMA 3.3. *We have*

$$(3.5) \quad Y_W^*(\mathcal{D}v, x)\alpha = -x^2 \frac{d}{dx} Y_W^*(v, x)\alpha$$

for $v \in V$, $\alpha \in W^*$.

PROOF. It is straightforward: For any $w \in W$, we have

$$\begin{aligned} \langle Y_W^*(\mathcal{D}v, x)\alpha, w \rangle &= \langle \alpha, Y_W(\mathcal{D}v, x^{-1})w \rangle = \frac{d}{dx^{-1}} \langle \alpha, Y_W(v, x^{-1})w \rangle \\ &= -x^2 \frac{d}{dx} \langle \alpha, Y_W(v, x^{-1})w \rangle = -x^2 \frac{d}{dx} \langle Y_W^*(v, x)\alpha, w \rangle, \end{aligned}$$

as needed. \square

DEFINITION 3.4. Let V be a weak quantum vertex algebra, (W, Y_W) a V -module. Define $D(W)$ to be the set of $\alpha \in W^*$ satisfying the condition

$$(3.6) \quad Y_W^*(v, x)\alpha \in W^*((x)) \quad \text{for every } v \in V.$$

It is clear that $D(W)$ is a subspace of W^* . Furthermore, we have:

THEOREM 3.5. *Let V be a weak quantum vertex algebra with a constant map \mathcal{S} and let (W, Y_W) be a V -module. Then*

$$(3.7) \quad Y_W^*(v, x)\alpha \in D(W)((x)) \quad \text{for } v \in V, \alpha \in D(W).$$

Furthermore, the pair $(D(W), Y_W^*)$ carries the structure of a ϕ -coordinated V° -module with $\phi(x, z) = \frac{x}{1+zx}$.

PROOF. Let $u, v \in V$, $\alpha \in D(W)$. From the \mathcal{S} -Jacobi identity for Y_W we have

$$\begin{aligned} &x_0^{-1} \delta \left(\frac{x_1^{-1} - x_2^{-1}}{x_0} \right) Y_W^*(v, x_2) Y_W^*(u, x_1) \alpha \\ &\quad - x_0^{-1} \delta \left(\frac{x_2^{-1} - x_1^{-1}}{-x_0} \right) \sum_{i=1}^r Y_W^*(u^{(i)}, x_1) Y_W^*(v^{(i)}, x_2) \alpha \\ (3.8) \quad &= x_2 \delta \left(\frac{x_1^{-1} - x_0}{x_2^{-1}} \right) Y_W^*(Y(u, x_0)v, x_2) \alpha, \end{aligned}$$

where $u^{(i)}, v^{(i)}$ ($i = 1, \dots, r$) are vectors in V such that (2.20) holds with $f_i(x) = 1$. Let $n \in \mathbb{Z}$. Applying $\text{Res}_{x_0} \text{Res}_{x_1} x_1^n$ to (3.8), we get

$$\begin{aligned} & Y_W^*(v, x_2) \text{Res}_{x_1} x_1^n Y_W^*(u, x_1) \alpha - \text{Res}_{x_1} x_1^n \sum_{i=1}^r Y_W^*(u^{(i)}, x_1) Y_W^*(v^{(i)}, x_2) \alpha \\ &= \text{Res}_{x_0} (x_2^{-1} + x_0)^{-n-2} Y_W^*(Y(u, x_0)v, x_2) \alpha \\ &= \sum_{j \geq 0} \binom{-n-2}{j} x_2^{n+2+j} Y_W^*(u_j v, x_2) \alpha. \end{aligned}$$

That is,

$$\begin{aligned} & Y_W^*(v, x_2) \text{Res}_{x_1} x_1^n Y_W^*(u, x_1) \alpha \\ &= \sum_{j \geq 0} \binom{-n-2}{j} x_2^{n+2+j} Y_W^*(u_j v, x_2) \alpha + \text{Res}_{x_1} x_1^n \sum_{i=1}^r Y_W^*(u^{(i)}, x_1) Y_W^*(v^{(i)}, x_2) \alpha. \end{aligned}$$

Since $\alpha \in D(W)$ and $u_j v = 0$ for j sufficiently large, we get

$$Y_W^*(v, x_2) \text{Res}_{x_1} x_1^n Y_W^*(u, x_1) \alpha \in W^*((x_2)).$$

This proves $Y_W^*(u, x) \alpha \in D(W)((x))$ for any $u \in V$, $n \in \mathbb{Z}$.

In the first part, we have proved that $Y_W^*(\cdot, x)$ gives rise to a linear map from V to $\text{Hom}(D(W), D(W)((x)))$. It is clear that $Y_W^*(\mathbf{1}, x) = 1_{W^*}$. Furthermore, let $u, v \in V$, $\alpha \in D(W)$. Assume that $\bar{u}^{(j)}, \bar{v}^{(j)} \in V$, $j = 1, \dots, s$ such that

$$Y(v, x)u = \sum_{j=1}^s e^{x\mathcal{D}} Y(\bar{u}^{(j)}, -x) \bar{v}^{(j)}.$$

Then using Lemma 3.3 we obtain

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_2^{-1} - x_1^{-1}}{x_0} \right) Y_W^*(u, x_1) Y_W^*(v, x_2) \alpha \\ & \quad - x_0^{-1} \delta \left(\frac{x_1^{-1} - x_2^{-1}}{-x_0} \right) \sum_{j=1}^s Y_W^*(\bar{u}^{(j)}, x_2) Y_W^*(\bar{v}^{(j)}, x_1) \alpha \\ &= x_1 \delta \left(\frac{x_2^{-1} - x_0}{x_1^{-1}} \right) Y_W^*(Y(v, x_0)u, x_1) \alpha \\ &= x_1 \delta \left(\frac{x_2^{-1} - x_0}{x_1^{-1}} \right) Y_W^*(e^{x_0 \mathcal{D}} Y^o(u, -x_0)v, x_1) \alpha \\ &= x_1 \delta \left(\frac{x_2^{-1} - x_0}{x_1^{-1}} \right) e^{-x_0 x_1^2 \frac{\partial}{\partial x_1}} Y_W^*(Y^o(u, -x_0)v, x_1) \alpha \\ &= x_1 \delta \left(\frac{x_2^{-1} - x_0}{x_1^{-1}} \right) Y_W^* \left(Y^o(u, -x_0)v, \frac{1}{x_1^{-1} + x_0} \right) \alpha \\ (3.9) \quad &= x_1 \delta \left(\frac{x_2^{-1} - x_0}{x_1^{-1}} \right) Y_W^*(Y^o(u, -x_0)v, x_2) \alpha, \end{aligned}$$

noticing that

$$e^{-zx^2 \frac{\partial}{\partial x}} x = \phi_1(x, -z) = \frac{x}{1 + zx} = \frac{1}{x^{-1} + z}.$$

Then by Proposition 3.1 we conclude that $(D(W), Y_W^*)$ carries the structure of a ϕ -coordinated V^o -module. \square

REMARK 3.6. Assume $\phi(x, z) = e^{zp(x)\frac{d}{dx}} \cdot x$ with $p(x) \in \mathbb{C}[x, x^{-1}]$. Let (W, Y_W) be a ϕ -coordinated V -module. We believe that it is also true that $(D(W), Y_W^*)$ is a ϕ^* -coordinated V^o -module with $\phi^*(x, z)$ defined as in Lemma 2.4. What we lack is a Jacobi identity for ϕ -coordinated V -modules.

REMARK 3.7. Let V be a vertex operator algebra and let (W, Y_W) be a module for V viewed as a vertex algebra. For $v \in V$, following [FHL] define $Y'_W(v, x) \in (\text{End} W^*)[[x, x^{-1}]]$ by

$$\langle Y'_W(v, x)\alpha, w \rangle = \langle \alpha, Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x)w \rangle$$

for $\alpha \in W^*$, $w \in W$. It can be readily seen that

$$D(W) = \{\alpha \in W^* \mid Y'_W(v, x)\alpha \in W^*((x)) \text{ for all } v \in V\}.$$

It was proved in [L2] that $(D(W), Y'_W)$ is a module for V viewed as a vertex algebra.

The following is an interpretation of Theorem 5.2.1 of [FHL] in terms of ϕ -coordinated modules:

PROPOSITION 3.8. *Let V be a vertex operator algebra and let W be a vector space equipped with a linear map*

$$Y_W(\cdot, x) : V \rightarrow \text{Hom}(W, W((x))) \subset (\text{End} W)[[x, x^{-1}]]$$

such that $Y_W(\mathbf{1}, x) = 1_W$. For $v \in V$, set

$$Y_W^{new}(v, x) = Y_W(e^{xL(1)}(-x^{-2})^{L(0)}v, x).$$

Then (W, Y_W) is a ϕ -coordinated V -module with $\phi(x, z) = \frac{x}{1+zx}$ if and only if (W, Y_W^{new}) is a module for V viewed as a vertex algebra. Furthermore, this gives rise to an isomorphism between the categories of ϕ -coordinated V -modules and V -modules with V viewed as a vertex algebra.

PROOF. It basically follows from the arguments in [FHL]. By Proposition 3.1, (W, Y_W) is a ϕ -coordinated V -module if and only if

$$\begin{aligned} z^{-1}\delta\left(\frac{x_2^{-1}-x_1^{-1}}{z}\right)Y_W(u, x_1)Y_W(v, x_2) - z^{-1}\delta\left(\frac{x_1^{-1}-x_2^{-1}}{-z}\right)Y_W(v, x_2)Y_W(u, x_1) \\ = x_2\delta\left(\frac{x_1^{-1}+z}{x_2^{-1}}\right)Y_W(Y(u, -z)v, x_2) \end{aligned}$$

for all $u, v \in V$, which is

$$\begin{aligned} z^{-1}\delta\left(\frac{x_1-x_2}{zx_1x_2}\right)Y_W(u, x_1)Y_W(v, x_2) - z^{-1}\delta\left(\frac{x_2-x_1}{-zx_1x_2}\right)Y_W(v, x_2)Y_W(u, x_1) \\ = x_2\delta\left(\frac{x_2+zx_1x_2}{x_1}\right)Y_W(Y(u, -z)v, x_2). \end{aligned}$$

Setting $x_0 = zx_1x_2$, we get

$$\begin{aligned} x_0^{-1}\delta\left(\frac{x_1-x_2}{x_0}\right)Y_W(u, x_1)Y_W(v, x_2) - x_0^{-1}\delta\left(\frac{x_2-x_1}{-x_0}\right)Y_W(v, x_2)Y_W(u, x_1) \\ = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W(Y(u, -x_0x_1^{-1}x_2^{-1})v, x_2) \\ (3.10) \quad = x_1^{-1}\delta\left(\frac{x_2+x_0}{x_1}\right)Y_W\left(Y\left(u, -\frac{x_0}{(x_2+x_0)x_2}\right)v, x_2\right) \end{aligned}$$

for all $u, v \in V$. Set

$$\Phi(x) = e^{xL(1)}(-x^{-2})^{L(0)}.$$

We see that (3.10) is equivalent to

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W(\Phi(x_1)u, x_1) Y_W(\Phi(x_2)v, x_2) \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W(\Phi(x_2)v, x_2) Y_W(\Phi(x_1)u, x_1) \\ & = x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) Y_W \left(Y \left(\Phi(x_1)u, -x_0 x_1^{-1} x_2^{-1} \right) \Phi(x_2)v, x_2 \right) \\ (3.11) \quad & = x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) Y_W \left(Y \left(\Phi(x_2 + x_0)u, -\frac{x_0}{(x_2 + x_0)x_2} \right) \Phi(x_2)v, x_2 \right) \end{aligned}$$

for all $u, v \in V$. The formula (5.2.36) in [FHL] states

$$(3.12) \quad Y \left(\Phi(x_2 + x_0)u, -\frac{x_0}{(x_2 + x_0)x_2} \right) \Phi(x_2)v = \Phi(x_2)Y(u, x_0)v.$$

Then (3.11) is equivalent to

$$\begin{aligned} & x_0^{-1} \delta \left(\frac{x_1 - x_2}{x_0} \right) Y_W^{new}(u, x_1) Y_W^{new}(v, x_2) \\ & - x_0^{-1} \delta \left(\frac{x_2 - x_1}{-x_0} \right) Y_W^{new}(v, x_2) Y_W^{new}(u, x_1) \\ & = x_1^{-1} \delta \left(\frac{x_2 + x_0}{x_1} \right) Y_W^{new}(Y(u, x_0)v, x_2) \end{aligned}$$

for all $u, v \in V$. Consequently, (W, Y_W) is a ϕ -coordinated V -module if and only if (W, Y_W^{new}) is a module for V viewed as a vertex algebra. The “furthermore” assertion is clear. \square

References

- [BK] B. Bakalov and V. Kac, Field algebras, *Internat. Math. Res. Notices* **3** (2003), 123-159.
- [B] R. E. Borcherds, Vertex algebras, in “*Topological Field Theory, Primitive Forms and Related Topics*” (Kyoto, 1996), edited by M. Kashiwara, A. Matsuo, K. Saito and I. Satake, Progress in Math. 160, Birkhäuser, Boston, 1998, pp 35-77.
- [D] C. Dong, Twisted modules for vertex algebras associated with even lattices, *J. Algebra* **165** (1994), 91-112.
- [EK] P. Etingof and D. Kazhdan, Quantization of Lie bialgebras, V, *Selecta Mathematica, New Series* **6** (2000), 105-130.
- [FFR] A. J. Feingold, I. B. Frenkel and J. F. Ries, *Spinor Construction of Vertex Operator Algebras, Triality, and $E_8^{(1)}$* , Contemporary Math. **121**, 1991.
- [FHL] I. B. Frenkel, Y.-Z. Huang and J. Lepowsky, *On Axiomatic Approaches to Vertex Operator Algebras and Modules*, *Memoirs Amer. Math. Soc.* **104**, Providence, RI, 1993.
- [FLM] I. Frenkel, J. Lepowsky and A. Murerman, *Vertex Operator Algebras and the Monster*, Pure and Applied Math., Vol. 134, Academic Press, Boston, 1988.
- [H] M. Hazewinkel, *Formal Groups and Applications*, Pure and Applied Math., Vol. 78, Academic Press, New York-San Francisco-London, 1978.
- [HLZ1] Y.-Z. Huang, J. Lepowsky, L. Zhang, Logarithmic tensor category theory for generalized modules for a conformal vertex algebra, I: Introduction and strongly graded algebras and their generalized modules, arXiv:1012.4193.
- [HLZ2] Y.-Z. Huang, J. Lepowsky, L. Zhang, Logarithmic tensor theory II, Logarithmic formal calculus and properties of logarithmic intertwining operators, arXiv:1012.4196.

- [Le] J. Lepowsky, Application of a ‘Jacobi identity’ for vertex operator algebra to Zeta values and differential operators, *Lett. Math. Phys.* **53** (2000), 87-103.
- [L1] H.-S. Li, Local systems of twisted vertex operators, vertex superalgebras and twisted modules, in: *Moonshine, the Monster and Related Topics*, Proc. Joint Summer Research Conference, Mount Holyoke, 1994, in: C. Dong, G. Mason (Eds.), *Contemp. Math.* **193**, Amer. Math. Soc., Providence, RI, 1996, pp 203-236.
- [L2] H.-S. Li, Some finiteness properties of regular vertex operator algebras, *J. Algebra* **212** (1999), 495-514.
- [L3] H.-S. Li, Nonlocal vertex algebras generated by formal vertex operators, *Selecta Mathematica, New Series* **11** (2005), 349-397.
- [L4] H.-S. Li, Constructing quantum vertex algebras, *Int. J. Math.* **17** (2006), 441-476.
- [L5] H.-S. Li, Vertex F -algebras and their ϕ -coordinated modules, *J. Pure Appl. Algebra* (2010).
- [L6] H.-S. Li, ϕ -coordinated quasi-modules for quantum vertex algebras, *Commun. Math. Phys.* **308** (2011), 703-741.
- [L7] H.-S. Li, G -equivariant ϕ -coordinated quasi-modules for quantum vertex algebras, *J. Math. Phys.* **54** (2013), 1-26.
- [LTW] H.-S. Li, S. Tan and Q. Wang, Twisted modules for quantum vertex algebras, *J. Pure Applied Algebra* **214** (2010), 201-220.
- [S] J. Sun, Contragredient modules and invariant bilinear forms on Möbius nonlocal vertex algebras, *Algebra Colloquium* **20** (2013), 403-416.
- [X1] X.-P. Xu, Intertwining operators for twisted modules of a colored vertex operator superalgebras, *J. Algebra* **175** (1995), 241-273.
- [X2] X.-P. Xu, *Introduction to Vertex Operator Superalgebras and Their Modules*, Mathematics and its Applications. Vol. **456**, Kluwer Academic Publishers, Dordrecht, 1998.
- [Z1] Y.-C. Zhu, Vertex operator algebras, elliptic functions and modular forms, Ph.D. thesis, Yale University, 1990.
- [Z2] Y.-C. Zhu, Modular invariance of characters of vertex operator algebras, *J. Amer. Math. Soc.* **9** (1996), 237-302.

DEPARTMENT OF MATHEMATICAL SCIENCES, RUTGERS UNIVERSITY, CAMDEN, NEW JERSEY 08102, AND SCHOOL OF MATHEMATICAL SCIENCES, XIAMEN UNIVERSITY, FUJIAN, CHINA
E-mail address: hli@camden.rutgers.edu